MFH4U – Unit 7: Exponential and Logarithmic Functions
Lesson 1: Exploring the Logarithmic Function

Recall: * To find the inverse of a function switch x and y and solve for y.
* A graph and its inverse are symmetrical about the line y = x.

From their invention by John Napier in the 17^{th} century until the development of computers and calculators, logs were the only effective tools for numerical computation in astronomy, chemistry, physics and engineering.

The 2 most important types of logs are those to base 10, called common logs, and those to base e, called natural logs. We will look at e later in the course.

Note: $\log_{10}$ is equivalent to log
$\log_e$ is equivalent to ln

A log function is the inverse of an exponential function.

$y = \log_a x$ is equivalent to $x = a^y$ where a > 0 and $a \neq 1$

ex. Write in exponential form.

a) $\log_2 32 = 5$
$32 = 2^5$

b) $\log 1 = 0$
$\log_{10} 1 = 0$
$1 = 10^0$

ex. Write in logarithmic form.

a) $\frac{1}{16} = 4^{-2}$
$\log_4 \frac{1}{16} = -2$

b) $3^4 = 81$
$\log_3 81 = 4$

ex. Solve

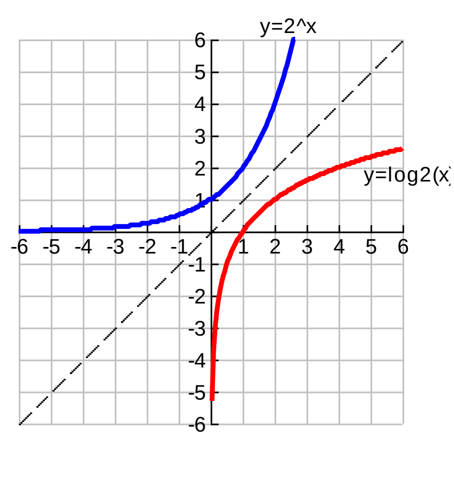
a) $x = \log_3 27$
$3^x = 27$
$3^x = 3^3$
$x = 3$

b) $\log_5 \frac{1}{125} = y$
$\frac{1}{125} = 5^y$
$5^{-3} = 5^y$
$y = -3$

ex. Graph the function and its inverse.
a) $y = \log_2 x$

Find inverse:
$x = \log_2 y$
$y = 2^x$

Domain: $\{x \in \mathbb{R}\}$ = range of $y = \log_2 x$
Range: $\{y \in \mathbb{R} \mid y > 0\}$ = domain of $y = \log_2 x$
y-int = 1 $\therefore$ $y = \log_2 x$ has an x-int of 1



Asymptote: y = 0 $\therefore$ $y = \log_2 x$ has a vertical asymptote at x = 0 and $y = 2^x$ has a horizontal asymptote at y = 0.

b) $y = -\log_2 x$

Find inverse:

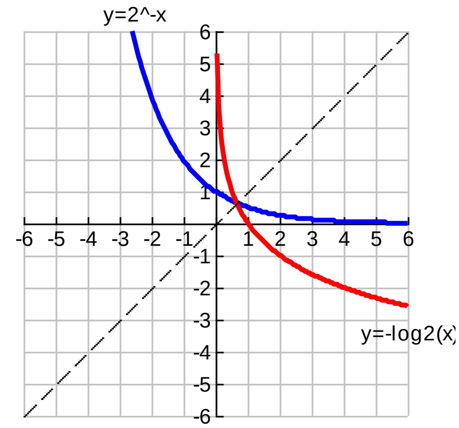
$$x = -\log_2 y$$
$$-x = \log_2 y$$
$$y = 2^{-x}$$

Domain: $\{x \in \mathbb{R}\}$

Range: $\{y \in \mathbb{R} \mid y > 0\}$

x-int of 1

VA: x = 0



c) $y = \log_2(-x)$

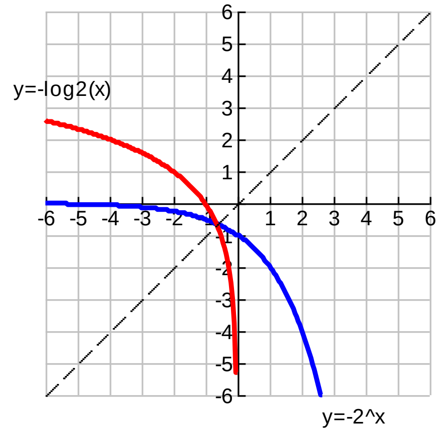
Inverse:

$$x = \log_2(-y)$$
$$-y = 2^x$$
$$y = -2^x$$

Domain: $\{x \in \mathbb{R}\}$

Range: $\{y \in \mathbb{R} \mid y < 0\}$

VA: x = 0



HW: Pg 451 #1ac, 2ac, 3, 6, 7, 8, 9

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Lesson 2: Transformations of Logarithmic Functions

What are the effects of each parameter in $y = a\log_{10}\left[k(x-d)\right]+c$ on the graph of $y=\log x$?

Use a graphing calculator to help you describe each transformation compared to $y=\log x$:

	Function	Transformation
a)	$y=\log(x+2)$	
b)	$y=\log(x-3)$	
c)	$y=\log x+4$	
d)	$y=\log x-2$	
e)	$y=3\log x$	
f)	$y=\frac{1}{2}\log x$	
g)	$y=-\log x$	
h)	$y=\log(2x)$	
i)	$y=\log(0.2x)$	
j)	$y=\log(-x)$	

In summary:

Given a logarithmic function of the form $f(x)=a\log_{10}\left[k(x-d)\right]+c$

- $|a|$ gives the vertical stretch/compression
- If $a < 0$ there is a reflection in the x-axis
- $|k|$ gives the horizontal stretch/compression
- If $k < 0$ there is a reflection in the y-axis
- d gives the horizontal translation
- c gives the vertical translation

The location of the vertical asymptote is dependent on the horizontal translation.

Given $y = a^x$ are there any values of a that make y something other than an exponential function?

- If a = 1 we get y = 1^x or y = 1 which is linear.
- If a = 0 we get y = 0^x or y = 0 except where x = 0 where the function is undefined (the graph will have a hole at x = 0).

What happens if *a* is negative?
ex. $y = (-2)^x$

x	-4	-3	-2	-1	0	1	2	3	4
y	0.0625	-0.125	0.25	-0.5	1	-2	4	-8	16

What if $x = \frac{1}{2}$?

$$y = (-2)^{\frac{1}{2}}$$
$$y = \sqrt{-2}$$
undefined

The graph would be discontinuous therefore we must restrict *a* to positive real numbers not equal to 1. Therefore the base of a logarithmic function must be a positive real number not equal to 1.

HW: Pg. 457 #4, 8, 9

Lesson 3: Evaluating Logarithms

Recall:

$y = \log_a x \;\Rightarrow\; x = a^y$

Solve each of the following:

a) $y = \log_2 2$

b) $y = \log_3 3^2$

c) $y = \log_4 4^7$

Solutions:

$a)\ y = \log_2 2$

$2^y = 2$

$y = 1$

$b)\ y = \log_3 3^2$

$3^y = 3^2$

$y = 2$

$c)\ y = \log_4 4^7$

$4^y = 4^7$

$y = 7$

Generally:

$\boxed{\log_a a^x = x}$

Evaluate each of the following:

a) $\log_5 1$

b) $\log_2 1$

Solutions:

$a)\ x = \log_5 1$

$5^x = 1$

$5^x = 5^0$

$x = 0$

$b)\ x = \log_2 1$

$2^x = 1$

$2^x = 2^0$

$x = 0$

Generally:

$\boxed{\log_a 1 = 0}$

Evaluate each of the following:

a) $2^{\log_2 x}$

b) $5^{\log_5 x}$

Solutions:

$a)\ 2^{\log_2 x}$

$let\ y = \log_2 x$

$\therefore x = 2^y$

$2^{\log_2 x} = 2^y$

$2^{\log_2 x} = x$

$b)\ 5^{\log_5 x}$

$let\ y = \log_5 x$

$\therefore x = 5^y$

$5^{\log_5 x} = 5^y$

$5^{\log_5 x} = x$

Generally:

$\boxed{a^{\log_a x} = x}$

In order to evaluate logarithms

- ✓ Let a variable equal the expression
- ✓ Rewrite in exponential form
- ✓ Express both sides of the equation as powers with the same base

Ex. Evaluate each of the following:

a) $\log_8 64$

b) $\log_7 49^4$

c) $\log 0.001$

d) $\log_{0.5} 32$

Solutions:

a) $\log_8 64$	b) $\log_7 49^4$
c) $\log 0.001$	d) $\log_{0.5} 32$

Ex. Solve $\log_x \frac{8}{27} = 3$

Ex. The altitude of an aircraft is a function of the outside air pressure. The higher the aircraft, the thinner the air around it, and the lower the pressure. When the outside pressure is P, the altitude (h), in miles, is given by $h = \frac{-100}{9}\log\frac{P}{B}$, where B is the atmospheric pressure at sea level. Determine the altitude of an airplane if the pressure at sea level is 30 inches of mercury and the outside air pressure is 18.2 inches of mercury.

$$
\begin{aligned}
h &= \frac{-100}{9}\log\frac{P}{B} \\
&= \frac{-100}{9}\log\frac{18.2}{30} \\
&\doteq \frac{-100}{9}(-0.217) \\
&\doteq 2.41
\end{aligned}
$$

The plane is approximately 2.41 miles high.

HW: Pg. 466 #3, 4abc, 5, 9, 10, 14a

Lesson 4: Laws of Logarithms

Recall:

$$\boxed{\log_a a^x = x}$$

1. Product Law

Given $\log_a m = b$ and $\log_a n = c$, determine $\log_a(mn)$.

$$\log_a m = b \Rightarrow m = a^b$$
$$\log_a n = c \Rightarrow n = a^c$$

$$\begin{aligned}\log_a(mn) &= \log_a\left(a^b \cdot a^c\right)\\ &= \log_a\left(a^{b+c}\right)\\ &= b + c\\ &= \log_a m + \log_a n\end{aligned}$$

$\therefore$ The logarithm of a product is the sum of the logarithms.

$$\boxed{\log_a(mn) = \log_a m + \log_a n}$$

2. Quotient Law

Given $\log_a m = b$ and $\log_a n = c$, determine $\log_a\left(\frac{m}{n}\right)$.

$$m = a^b \;\&\; n = a^c$$

$$\begin{aligned}\log_a\left(\frac{m}{n}\right) &= \log_a\left(\frac{a^b}{a^c}\right)\\ &= \log_a\left(a^{b-c}\right)\\ &= b - c\\ &= \log_a m - \log_a n\end{aligned}$$

$\therefore$ The logarithm of a quotient is the difference of the logarithms.

$$\boxed{\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n}$$

3. Power Law

Given $\log_a m = b$, determine $\log_a m^p$.

$$m = a^b$$

$$\begin{aligned}\log_a m^p &= \log_a\left(a^b\right)^p\\ &= \log_a\left(a^{bp}\right)\\ &= bp\\ &= p \cdot \log_a m\end{aligned}$$

$\therefore$ The logarithm of a power is equal to the exponent multiplied by the logarithm of the base.

$$\boxed{\log_a m^p = p \cdot \log_a m}$$

Ex. Rewrite $\log 12 + \frac{1}{2}\log 7 - \log 2$ as a single logarithm.

$$\begin{aligned}\log 12 + \frac{1}{2}\log 7 - \log 2 &= \log 12 + \log 7^{\frac{1}{2}} - \log 2\\ &= \log 12 + \log\sqrt{7} - \log 2\\ &= \log\left(\frac{12\sqrt{7}}{2}\right)\\ &= \log\left(6\sqrt{7}\right)\end{aligned}$$

Ex. Evaluate $\log 30 + \log\left(\frac{10}{3}\right)$ without a calculator.

$$\begin{aligned}\log 30 + \log\left(\frac{10}{3}\right) &= \log\left(\frac{30 \cdot 10}{3}\right)\\ &= \log 100\\ &= \log 10^2\\ &= 2\end{aligned}$$

HW: Pg. 475 #2abf, 3ad, 4aef, 6f, 9ef, 10f, 11cf

Lesson 5: Solving Exponential and Logarithmic Equations

Recall:

$\boxed{\log_a a^x = x}$

$\boxed{\log_a(mn) = \log_a m + \log_a n}$

$\boxed{\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n}$

$\boxed{\log_a m^p = p \cdot \log_a m}$

The definition and properties of logarithms can be used to solve equations in which either powers or logarithms appear. If the unknown occurs in an exponent then the strategy is to isolate it by taking the logarithm of both sides.

Ex. Solve $3^{x+2} = 4$

→ Take log (base 10) of each side then use the laws of logarithms to simplify:

$$\begin{aligned} 3^{x+2} &= 4 \\ \log\left(3^{x+2}\right) &= \log 4 \\ (x+2)\log 3 &= \log 4 \\ x+2 &= \frac{\log 4}{\log 3} \\ x &= \frac{\log 4}{\log 3} - 2 \\ x &\doteq -0.7381 \end{aligned}$$

→ It is always advisable to check your solution(s):

$$\begin{aligned} LS &= 3^{x+2} \\ &\doteq 3^{-0.7381+2} \\ &\doteq 3^{1.2619} \\ &\doteq 4.00018 \end{aligned} \qquad RS = 4$$

LS = RS

$\therefore\ x \doteq -0.74$

Ex. Evaluate $\log_4 17$

→ Most calculators cannot evaluate logs to bases other than 10 or e.

→ Rewrite in exponential form then take log of both sides and solve:

$$
\begin{aligned}
let\ x &= \log_4 17 \\
4^x &= 17 \\
\log 4^x &= \log 17 \\
x \cdot \log 4 &= \log 17 \\
x &= \frac{\log 17}{\log 4} \\
x &\doteq 2.04
\end{aligned}
$$

Ex. Solve $\log_2 x - \log_2 3 = \log_2 6$ in two different ways.

Solution #1:	Solution #2:
$\log_2 x - \log_2 3 = \log_2 6$ $\log_2 x = \log_2 6 + \log_2 3$ $\log_2 x = \log_2 (6 \cdot 3)$ $\log_2 x = \log_2 18$ $x = 18$	$\log_2 x - \log_2 3 = \log_2 6$ $\log_2 \left(\frac{x}{3}\right) = \log_2 6$ $\frac{x}{3} = 6$ $x = 18$

Ex. Solve $\log(x+2) + \log(x-1) = 1$

→ Rewrite each side as a single logarithm (with the same base) then solve the resulting equation.

$$
\begin{aligned}
\log(x+2) + \log(x-1) &= 1 \\
\log\left[(x+2)(x-1)\right] &= 1 \\
\log\left(x^2 + x - 2\right) &= \log_{10} 10 \\
x^2 + x - 2 &= 10 \\
x^2 + x - 12 &= 0 \\
(x+4)(x-3) &= 0 \\
\therefore x = -4 \ &or\ x = 3
\end{aligned}
$$

→ Check your solutions:

If x = 3	If x = -4
$LS = \log(x+2) + \log(x-1)$ $\quad RS = 1$ $= \log(3+2) + \log(3-1)$ $= \log 5 + \log 2$ $= \log(5 \cdot 2)$ $= \log 10$ $= \log_{10} 10^1$ $= 1$ LS = RS	$LS = \log(x+2) + \log(x-1)$ $\quad RS = 1$ $= \log(-4+2) + \log(-4-1)$ $= \log(-2) + \log(-5)$ Since the log of a negative number is not defined, -4 is not a root.

$\therefore x$ = 3

It is always your responsibility to check all solutions!

Ex. Solve $6^{3x} = 4^{2x-3}$

$$\begin{aligned}
6^{3x} &= 4^{2x-3} \\
\log 6^{3x} &= \log 4^{2x-3} \\
3x\log 6 &= (2x-3)\log 4 \\
3x\log 6 &= 2x\log 4 - 3\log 4 \\
3x\log 6 - 2x\log 4 &= -3\log 4 \\
x(3\log 6 - 2\log 4) &= -3\log 4 \\
x &= \frac{-3\log 4}{3\log 6 - 2\log 4} \\
x &\doteq -1.6
\end{aligned}$$

Ex. The power source used by satellites is called a radioisotope. The power output of the radioisotope is given by the equation $P = 50(0.996)^t$, where P is the power in watts and t is the time in years. If the equipment in the satellite needs at least 15 W of power to function, for how long can the satellite operate before needing recharging?

$\rightarrow$ Determine t when P = 15

$$\begin{aligned}
P &= 50(0.996)^t \\
15 &= 50(0.996)^t \\
\frac{15}{50} &= 0.996^t \\
0.3 &= 0.996^t \\
\log 0.3 &= \log 0.996^t \\
\log 0.3 &= t \cdot \log 0.996 \\
t &= \frac{\log 0.3}{\log 0.996} \\
t &\doteq 300.39
\end{aligned}$$

$\therefore$ In theory, the satellite can operate for about 300 years.

HW: Pg. 491 #4abcd, 5cf, 7acf, 12
Pg. 486 #10, 8a, 17a

MFH4U – Unit 7: Exponential and Logarithmic Functions
Lesson 6: Applications with Logarithms

Applications to Natural Sciences

Ex. $t = c \cdot \log_2 \dfrac{b(a-x)}{a(b-x)}$ is used to study certain chemical reactions

$\rightarrow$ x is the concentration of a substance at time t and a, b and c are constants

Ex. Exponential growth and decay: $m(t) = c\left(\frac{1}{2}\right)^{\frac{t}{h}}$

$\rightarrow$ $m(t)$ is the mass of a substance at time t and h is the half-life

Ex. pH scale: $pH = -\log(\text{concentration of } H^+)$

(see example on page 495)

Applications to Compound Interest

$A = P(1+i)^n$

$\rightarrow$ A is the amount of the loan or investment
$\rightarrow$ P is the principal
$\rightarrow$ i is the rate of interest per conversion period
$\rightarrow$ n is the number of conversion periods

Ex. Determine the length of time required for an investment of \$1000 to amount to \$1500 at a rate of 9% per year compounded quarterly.

$\rightarrow$ P = 1000, A = 1500, $i = \frac{0.09}{4} = 0.0225$; determine n

$$
\begin{aligned}
A &= P(1+i)^n \\
1500 &= 1000(1+0.0225)^n \\
1.5 &= 1.0225^n \\
\log 1.5 &= \log 1.0225^n \\
\log 1.5 &= n \log 1.0225 \\
n &= \frac{\log 1.5}{\log 1.0225} \\
n &\doteq 18.2
\end{aligned}
$$

The investment will amount to \$1500 during the 19th conversion period $\therefore$ 4.75 years

Logarithmic Scales

When quantities can vary over very large ranges it is sometimes convenient to take their logarithms in order to get a more manageable set of numbers.

Ex. Gapminder.org

Ex. The Richter scale which is used to measure the intensity of earthquakes. The ratio of intensities of the smallest recorded earthquake and largest recorded earthquake is approximately 800000000. The Richter scale gives much more manageable numbers to work with.

The magnitude, M, of an earthquake is $M = \log\left(\frac{I}{I_0}\right)$ where $\frac{I}{I_0}$ is the ratio of intensities between the earthquake being measured and that of a standard, low-level earthquake. You do not need to know either of these values as we work with the value of the magnitude.

An earthquake of magnitude 7 is 10 times as strong as an earthquake of magnitude 6.

a) How many times as intense as a standard earthquake is one measuring 2.4 on the Richter scale?

$$M = \log\left(\frac{I}{I_0}\right)$$
$$2.4 = \log\left(\frac{I}{I_0}\right)$$
$$10^{2.4} = \frac{I}{I_0}$$
$$\frac{I}{I_0} \doteq 251.2$$

It is approximately 251 times as intense.

b) What is the magnitude of an earthquake 1000 times as intense as a standard earthquake?

$$\begin{aligned} M &= \log\left(\frac{I}{I_0}\right) \\ &= \log 1000 \\ &= \log 10^3 \\ &= 3 \end{aligned}$$

It has magnitude 3.

c) How many times as intense was the 1964 Alaska earthquake of magnitude 8.5 compared to a moderately destructive earthquake of magnitude 6.0?

Magnitude 8.5	Magnitude 6.0
$M = \log\left(\frac{I}{I_0}\right)$	$M = \log\left(\frac{I}{I_0}\right)$
$8.5 = \log\left(\frac{I}{I_0}\right)$	$6 = \log\left(\frac{I}{I_0}\right)$
$\frac{I}{I_0} = 10^{8.5}$	$\frac{I}{I_0} = 10^{6}$

Ratio of intensities:

$$\begin{aligned} \frac{10^{8.5}}{10^6} &= 10^{2.5} \\ &\doteq 316 \end{aligned}$$

The Alaska earthquake was approximately 316 times as intense as the moderate earthquake.

Ex. A similar scheme is used in the decibel scale which measures the loudness of sounds.

The loudness, L (in decibels), is related to the sound's intensity, I (in watts per m^2) as follows:

$L = 10\log\frac{I}{I_0}$, where I_0 is the loudness of a barely audible sound – usually 10^{-12} W/m^2.

See example on page 498.

HW: Pg. 499 #1, 2, 3, 8, 10, 13, 14, 15

MHF4U – Unit 7

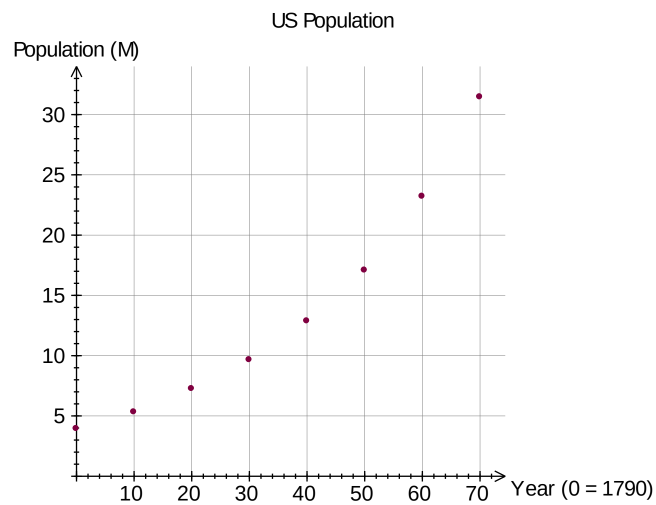
Lesson 7 – Exponential and Logarithmic Models

We create models to help predict values. Models can be created by observing patterns (common differences, common ratios) or using graphing technology.

ex. In the years before the Civil War, the population of the US grew rapidly, as shown in the following table. Find a model for this growth.

Year	1790	1800	1810	1820	1830	1840	1850	1860
Population in millions	3.93	5.31	7.24	9.64	12.86	17.07	23.19	31.44

Make 1790 year 0 and enter this new year data and the population data in a Lists & Spreadsheet page. Set up a scatter plot (G&G page) and look at the shape of the plot.



The shape suggests either a polynomial graph of even degree or an exponential graph. Populations generally grow exponentially so we should check for a common ratio to determine if it this case is exponential.

$\frac{5.31}{3.93} =$						

Since the ratios are almost constant, as they would be in an exponential model, we can find an exponential equation to model this data.

We will use the model $y = c \cdot a^x$, where 'c' is the initial value and 'a' is the growth factor.

Our common ratio is somewhere between 1.327 and 1.363, say 1.35. To find a value for 'a' we must take into account the fact that the data points are 10 years apart. This means that to get from the first data point to the second data point you would have to multiply by a 10 times, i.e.

$$3.93 \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a = 5.31$$

$$3.93 \cdot a^{10} = 5.31$$

Therefore to find 'a' we need the 10^{th} root of 1.35.

$$1.35^{\frac{1}{10}} \doteq 1.030465312$$

Store this value in your calculator!

Our model is now $y = c \cdot 1.03^x$.

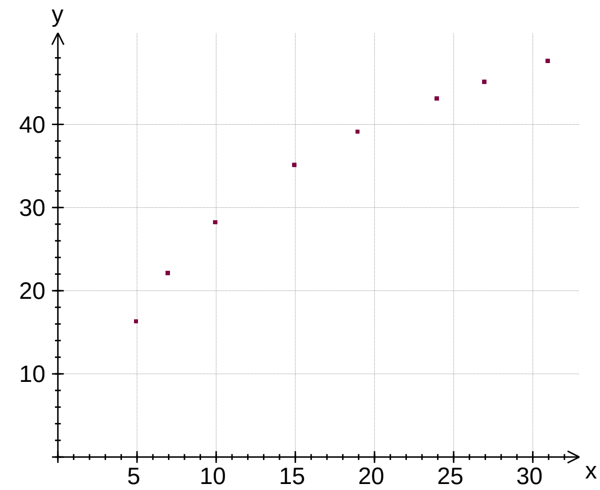
Choosing one data point for x and y, calculate the value of 'c'. Using (60,23.19), c = 3.94 so our model is y = 3.94(1.03)^x. Using a graphing calculator we can see how well our model fits the data.

The graphing calculator can also find an equation for us:

ex. Determine a logarithmic model for the following data.

x	5	7	10	15	19	24	27	31
y	16.2	22	28.1	35	39	43	45	47.5

Enter the data and plot them.



To determine if a logarithmic model is appropriate you can <u>linearize</u> the data. If 'y' represents a logarithmic function, taking the logarithm of the data in 'x' will create a linear relationship. If the graph of the new plot appears to be linear then a logarithmic model is a good choice for this data.

You need to plot log(x) vs y:
Make the 3rd column equal to log(1st column) then plot "L3" vs. "L2".
Does the graph look linear? If so, perform a linear regression on it and look at the correlation coefficient to see how well it fits the data.
Linear equation: ___________________________

Since we replaced x with log x, you need to replace x with log x in your linear equation. The result is the logarithmic equation that fits the data in "L1" & "L2".

Logarithmic equation:__________________________.

HW: pg 500 #11: model, 12abc; pg 507 #7b